Homotopy type theory : working invariantly in homotopy theory

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Homotopy theory

Homotopy theory is the study of homotopy types, i.e.,

- topological spaces up to weak homotopy equivalences, or
- CW-complexes up to homotopy equivalences, or
- simplicial sets up to weak equivalences.

It is distinct from topology in that we are only interested in homotopy-invariant properties and in non-pathologic spaces.
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Univalent Foundations

The primitive objects of mathematics behave like homotopy types:

- Two natural numbers are either equal, or different
- Two sets can be in bijection in several different ways
- Two categories can be equivalent in several different ways, and those equivalences can be naturally isomorphic is several different ways

The Univalent Foundations (UF), introduced by Vladimir Voevodsky, is an approach to the foundations of mathematics based on this idea.

- It is implemented in a variant of Martin–Löf type theory, well known in theoretical computer science and constructive mathematics,
- It enables us to formally check the correctness of proofs,
- It can be used to formalize all of mathematics (constructive by default, but compatible with classical logic)

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Invariant homotopy theory

UF can be used to formalize all of mathematics, and homotopy theory is part of mathematics. So we could formalize homotopy theory in UF:

- Define topological spaces using UF's notion of set and proposition
- Prove all usual theorems about homotopy theory
- Everything would basically works as expected

But invariant/synthetic homotopy theory is very different. It's using the core connection of UF with homotopy theory to reason *directly* about homotopy types. In particular:

- It is not the study of topological spaces (or simplicial sets)
- All concepts used in definitions/constructions/proofs are homotopy-invariant, as there is no underlying topological space.

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Examples of concepts which are not homotopy-invariant

- The notion of subspace
  e.g. the complement of a point or of a knot
- The notion of a map $f : E \to B$ being a fibration
  e.g. $\exp : \mathbb{R} \to \mathbb{S}^1$ homotopic to a constant map
- Quotients with respect to an equivalence relation
  e.g. projective spaces
- The usual definitions of matrix groups
  e.g. $\mathrm{SO}(n)$, grassmanians
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Invariant concepts

- Function spaces
- Path spaces
- Homotopy limits and homotopy colimits
  give many nice cell complexes, e.g. $\mathbb{S}^n$, $\mathbb{R}\mathrm{P}^n$ (Buchholtz, Rijke)
- Truncations
  give for instance π_k, $K(G, n)$
- Universes ("the space of all spaces", module size issues)

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Fibrations and the univalence axiom

Intuition

A fibration is a family of spaces parametrized by another space.

A fibration over B is a map $P : B \to \mathcal{U}$, where $\mathcal{U}$ is a universe, and its fibers are the $P(x)$, for x in B.

If B is defined as a cell complex/homotopy colimit, we define such a map by giving the images of all of the cells. In particular we need:

Univalence axiom (Voevodsky)

A path in the universe is the same thing as a homotopy equivalence between its endpoints.

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The universal cover of the circle

Definition

The circle $\mathbb{S}^1$ is generated by

$$b : \mathbb{S}^1,$$
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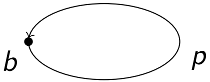
$$P : \mathbb{S}^1 \to \mathcal{U},$$
$$P(b) := \mathbb{Z},$$
$$P(p) := \begin{cases} \mathbb{Z} \to \mathbb{Z} \\ n \mapsto n+1 \end{cases}.$$



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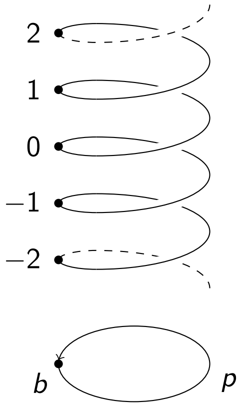

Some results

$\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$

- universal cover of $\mathbb{S}^1$ (Shulman)
- Hopf fibration
- James construction
- Blakers–Massey theorem (Favonia, Finster, Licata, Lumsdaine)
- cohomology
- Hopf invariant
- Thom isomorphism

covering spaces (Favonia)

Seifert–van Kampen theorem (Favonia, Shulman)

quaternionic Hopf fibration (Buchholtz)

Steenrod squares

The Steenrod squares are operations

$$\mathrm{Sq}^i : H^n(A, \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(A, \mathbb{Z}/2\mathbb{Z}).$$

In order to define the Steenrod squares invariantly:

- Define $\mathcal{U}_2$ to be the type of all *types-with-exactly-two-elements*,
- Prove that the cup product is homotopy-commutative in the sense that for all $X : \mathcal{U}_2$, there is a map

$$\smile_X : H^n(A, \mathbb{Z}/2\mathbb{Z})^X \to H^{2n}(A, \mathbb{Z}/2\mathbb{Z}),$$

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Future directions

- Do more homotopy theory invariantly, e.g., grassmanians, Bott periodicity, K-theory, spectral sequences, etc.
- Understand better the constructivity properties of homotopy type theory
- Work on the Agda library for homotopy type theory

Don't hesitate to talk to me if you want to know more, or if you know some homotopy theory that could benefit from this approach.

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