Experiments in cubical type theory

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Canonicity of Martin–Löf type theory

Martin–Löf type theory has the following properties

- *Strong normalisation*
  There is no infinite sequence of reductions.
- *Confluence*
  Two sequences of reductions starting from the same term can be made to converge again.
- *Subject reduction*
  Reduction does not change the type
- *Characterization of normal forms*
  Every normal form (term which cannot be reduced anymore) has a specific form which depends on its type.

In particular, every term t of type $\mathbb{N}$ reduces to a unique numeral (0, 1, 2, etc.).

Homotopy canonicity

When we add the univalence axiom (or any axiom), the property of canonicity fails. There can be other terms of type $\mathbb{N}$ which are stuck but are not numerals. For instance given any (explicit) equivalence $f : \mathbb{N} \simeq \mathbb{N}$, the following term is stuck:

$$\mathtt{coe}(\mathtt{ua}(f))(0) : \mathbb{N}$$

(where $\mathtt{coe}(p) : A \to B$ when $p : A = B$)

Conjecture (Voevodsky)

Given a term $t : \mathbb{N}$ constructed using the univalence axiom, we can construct two terms $u : \mathbb{N}$ and $p : t =_{\mathbb{N}} u$ such that u does not involve the univalence axiom.

Cubical type theory

- Cubical type theory is a different type theory, which has good computational properties (Huber, 2016), and in which we can prove the univalence axiom.
- There are additional definitional equalities, like

$$\mathrm{ap}_{g \circ f}(p) \equiv \mathrm{ap}_g(\mathrm{ap}_f(p))$$

- Some definitional equalities are not valid anymore, like

$$\mathtt{coe}(\mathtt{refl}_A)(a) \equiv a$$

- This does not solve the homotopy canonicity conjecture, but it is an answer to the same underlying problem: the non-constructivity of the univalence axiom.

Implementations of cubical type theory

- The paper describing cubical type theory is (Cohen, Coquand, Huber, Mörtberg, 2016) : "Cubical Type Theory: a constructive interpretation of the univalence axiom" (referred to as *CCHM*).
  It is implemented in a prototype proof assistant called `cubicaltt`[1].
- A former version was `cubical`[2], based on (Bezem, Coquand, Huber, 2014): "A cubical set model of type theory".

[1] `https://github.com/mortberg/cubicaltt/`
[2] `https://github.com/simhu/cubical`

Other cubical type theories

- Computational higher type theory (Angiuli, Favonia, Harper, 2017) is based on computational type theory à la NuPRL (implementation in progress in RedPRL[3])
- Cartesian cubical type theory (Angiuli, Brunerie, Coquand, Favonia, Harper, Licata, 2017) is another version of cubical type theory and has a few technical differences with CCHM (no connections, but the Kan compositions are more general).
- Work is in progress for implementing CCHM cubical type theory in Agda, by Andrea Vezzosi.[4]

[3] `https://github.com/RedPRL/sml-redprl`

[4] `https://agda.readthedocs.io/en/latest/language/cubical.html`

Test case: $\pi_4(\mathbb{S}^3)$

In order to test cubical type theory we need good test cases, one is the following:

Proposition (Brunerie, 2013)

There is a natural number n such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$.

Classically, we know that n is supposed to be equal to 2, but that fact was proved in HoTT only in 2016.
Therefore, the following is supposed to be true, but we would like to actually check it.

Goal

If we implement the definition of n in `cubicaltt` and normalize it, we should obtain 2.

Timeline

- 2013: Informal proof of the existence of the natural number n such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$
- 2015: First attempt to implement the definition of n in `cubical` (predecessor of `cubicaltt`), with Thierry Coquand and Simon Huber
- June 2017: New attempt in `cubicaltt` with the MRC group in Snowbird (with Vikraman Choudhury, Paul Gustafson, Dan Licata, Ian Orton, and Jon Sterling)
- end of 2017: Further simplifications, with Anders Mörtberg

Still no completed computation!

Overview of the definition

We want to compute the image of 1 by the following composition.

$$\begin{array}{l}\mathbb{Z}\\ \downarrow\\ \Omega^3\mathbb{S}^3 \xrightarrow{f} \Omega^3\mathbb{S}^2 \xrightarrow{g} \Omega^3\mathbb{S}^3 \xrightarrow{h} \|\Omega^2\mathbb{S}^2\|_0 \longrightarrow \Omega\mathbb{S}^1 \longrightarrow \mathbb{Z}\end{array}$$

- f: does not use univalence, but quite technical
- g: uses univalence in a very complicated way
- h: uses univalence three times
- The last two maps also use univalence.

Does not compute yet (takes too much time and memory), but we can try some subparts.

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Issue: Complexity of g

Sketch of the definition of g:

- Define the Hopf fibration $H : \mathbb{S}^2 \to \mathrm{Type}$ using univalence and the multiplication structure on the circle.
- Loop it three times to get a fibration $H^{\Omega^3} : \Omega^3\mathbb{S}^2 \to \mathrm{Type}$.
- The fibers of H are merely equivalent to $\mathbb{S}^1$ which is a 1-type, therefore the fibers of H^{Ω^3} are contractible.
- The total space of H is $\mathbb{S}^3$, therefore the total space of H^{Ω^3} is $\Omega^3\mathbb{S}^3$.
- Therefore we get a map $\Omega^3\mathbb{S}^2 \to \Omega^3\mathbb{S}^3$.

A different definition of the looping of a fibration (using `transport` instead of dependent paths) simplified the third step a lot, but makes the fourth step more complicated.

Issue: Regularity

In the version of `cubicaltt` used during the MRC, the term obtained after f was 5,000,000 characters long, mainly due to the lack of regularity and the presence of countless *empty systems*. In the last version of `cubicaltt` this has been somehow fixed, it is now only 100,000 characters long.

Issue: Definition of the spheres

How do we define the spheres?

We can define the spheres either by iterated suspension, by join with the booleans, or directly with a constructor of higher dimension.

Defining them directly is better (it divides the size of the term mentioned before by 10), but makes some definitions more complicated.

Issue: Which proofs are irrelevant?

In HoTT, h-propositions are not necessarily erasable, they may contain essential computational content.

It seems that `propIsEquiv` sometimes needs to be reduced (but not always), and it creates problems.

Same for `groupoidSET` and `twogroupoidGROUPOID`, can we (should we?) put them opaque?

Demo

$$\begin{array}{ccccccccccc} \color{red}{\mathbb{Z}} & & \color{red}{\mathbb{Z}} & & \color{green}{\mathbb{Z}} & & \color{green}{\mathbb{Z}} & & \color{green}{\mathbb{Z}} & & \\ \color{red}{\downarrow} & & \color{red}{\downarrow} & & \color{green}{\downarrow} & & \color{green}{\downarrow} & & \color{green}{\downarrow} & & \\ \Omega^3\mathbb{S}^3 & \longrightarrow & \Omega^2\|\mathbb{S}^2\|_2 & \longrightarrow & \Omega\|\Omega\mathbb{S}^2\|_1 & \longrightarrow & \|\Omega^2\mathbb{S}^2\|_0 & \longrightarrow & \Omega\mathbb{S}^1 & \longrightarrow & \mathbb{Z} \end{array}$$